Threshold Density for Fixed Energy Sandpile Model

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$$\begin{tikzcd}
D_0 \sim B_0 + e_0 s \arrow[r, dashed] \arrow[d, "v_1"'] & C_0 \arrow[d, "v_1"] \\
D_1 \sim B_1 + e_1 s \arrow[r, dashed] \arrow[d, phantom, "\vdots"] & C_1 \arrow[d, phantom, "\vdots"] \\
{} \arrow[d, "v_t"'] & {} \arrow[d, "v_t"] \\
D_t \sim B_t + e_t s \arrow[r, dashed] & C_t
\end{tikzcd}$$

Fixed Energy Model | Abelian Sandpile

We need to keep track of v_i & B_i

v_i: step | B_i: recurrent sandpile.

$$X_t = (v_t, B_t)$$

$$\mathbb{P}\left(X_{t+1} = (v, B) \mid X_t = (v', B')\right)$$

$$= \begin{cases} \alpha(v) & \text{if } B' \xrightarrow{(v')} B \\ 0 & \text{o.w.} \end{cases}$$

X_t is a HMC

Let $\pi(v, B) = \dfrac{\alpha(v)}{|S(G)|}$

$$\sum_{(v,B)\in \mathcal{V}} \pi(v', B')\, P\big((v', B'), (v, B)\big)$$

$$= \sum_{(v,B)\in \mathcal{V}} \frac{\alpha(v')}{|S(G)|}\, \alpha(v)\, \mathbb{1}_{\{B' \xrightarrow{(v')} B\}}$$

$$= \sum_{v \in V} \frac{\alpha(v')}{|S(G)|}\, \alpha(v) = \frac{\alpha(v)}{|S(G)|} = \pi(v, B)$$

Detour: Markov Renewal Theorem

$\mathcal{E} := \{(x,y) : P(x,y) > 0\}$ — digraph

$\ell : \mathcal{E} \longrightarrow \mathbb{N} \cup \{0\}$ — length function

$\lambda(P) = \sum_{e \in P} \ell(e)$ where P is a path in $\mathcal{E}$.

Aperiodic : gcd of lengths of closed paths is 1.

$\tau_d := \min\{t : \lambda_t \geq d\}$ where $\lambda_t := \sum_{i=1}^{t} \ell((X_{i-1}, X_i))$

↳ time at which length elapsed becomes greater than d

Thm: Let $x_0, x, y \in \Omega$, $e \in \mathbb{N}$

$$\lim_{d \to \infty} \mathbb{P}_{x_0}\left((X_{\tau_d - 1}, X_{\tau_d}, \lambda_{\tau_d} - d) = (x, y, e)\right)$$

$$= \begin{cases} \frac{1}{Z_\ell} \pi(x) P(x,y) & \text{if } 0 \leq e \leq \ell(x,y) \\ 0 & \text{o.w.} \end{cases}$$

$$Z_\ell := \sum_{x,y} \pi(x) P(x,y) \ell(x,y)$$

Proof follows from Kesten's general Markov Renewal.

Thm: G: Eulerian $\alpha : V \to (0,1]$

D_t : F.E. sandpile

τ : threshold

v_τ : epicentre

As $\deg(D_0) \to \infty$,

$$\mathbb{P}_{D_0}\Big((v_\tau, B_\tau, e_\tau) = (v, B, e) \Big) \to \begin{cases} \dfrac{\alpha(v)}{|S(G)|} & 0 \le e \le \beta_v(B) \\ 0 & \text{o.w.} \end{cases}$$

Pf:
$$\ell\big((v', B'), (v, B) \big) := \deg(B') - \deg(B) + 1$$
$$= \beta_v(B)$$

$$\tau = \min\{ t \geq 0 : e_t \geq 0 \}$$

$$\begin{aligned} e_t &= e_{t-1} + \beta_{v_t}(B_t) \\ &= e_{t-1} + \ell\big((v_{t-1}, B_{t-1}), (v_t, B_t) \big) \\ &= e_0 + \sum \ell\big((v_i, B_i), (v_{i+1}, B_{i+1}) \big) \\ &= e_0 + \lambda_t \end{aligned}$$

$$\tau = \tau_{e_0} = \min\{ t : \lambda_t \geq -e_0 \}$$

$\ell\big((s, B), (s, B) \big) = 1 \Rightarrow \ell$ is aperiodic.

$$\therefore \mathbb{P}_{D_\gamma}\Big(\big(v_{\tau-1}, B_{\tau-1}\big) = (v', B'), \big(v_\tau, B_\tau\big) = (v, B), e_\tau = e\Big)$$

$$\downarrow$$

$$\begin{cases} \dfrac{1}{Z}\dfrac{\alpha(v')}{|S(G)|}\underbrace{\alpha(v) \qquad B \xrightarrow{v} B}_{\text{from } F(x,y)} & \& \ 0 \leq e < \beta_v(B) \\ 0 & \text{o.w.} \end{cases}$$

$$Z = \sum_{((v',B'),(v,B)) \in \mathcal{E}} \frac{\alpha(v')}{|S(G)|}\alpha(v)\,\beta_v(B)$$

$$= \sum_{(v',B')} \frac{\alpha(v')}{|S(G)|} \sum_{v \in V} \alpha(v)\,\beta_v(B' * g_{v'})$$

$$= \sum_{v \in V} \frac{\alpha(v)}{|S(G)|} \sum_{(v',B') \in V} \alpha(v')\,\beta_v(B' * g_{v'})$$

$$= \sum_{v \in V} \frac{\alpha(v)}{|S(G)|} \sum_{v' \in V} \alpha(v') \sum_{B' \in S(G)} \beta_v(B' * g_{v'})$$

$$= \sum_{v', v \in V} \frac{\alpha(v)}{|S(G)|} \alpha(v') \sum_{B \in S(G)} \beta_v(B)$$

$$\sum_{B\in S(G)} \beta_v(B) = \sum_{B\in S(G)} \deg(B * g_v^{-1}) - \sum_{B\in S(G)} \deg(B) + \sum_{B\in S(G)} 1$$

$$= |S(G)|$$

$$\therefore \ Z = \sum_{v',v} \alpha(v)\,\alpha(v')$$

$$=1$$

sum over this

$$\mathbb{P}_{D_0}\big((v_{\tau-1}, B_{\tau-1}) = (v', B'), (v_\tau, B_\tau) = (v, B), e_\tau = e \big)$$

$$\downarrow$$

$$\begin{cases} \dfrac{1}{Z}\dfrac{\alpha(v')}{|S(G)|}\alpha(v) & B \xrightarrow{v} B \ \& \ 0 \le e < \beta_v(B) \\[2ex] 0 & \text{o.w.} \end{cases}$$

$$\deg(D_0) = \deg(B_0) + e_0$$

$$\deg(B_0) \geq 0$$

$$\deg(D_0) \geq e_0$$

$$\therefore \ \deg D_0 \to -\infty \ \Rightarrow \ e_0 \longrightarrow -\infty$$

Concludes the proof

$\square$

Corollary: $\mathbb{P}_{D_0}(B_\tau = B) \longrightarrow \frac{1}{|S(G)|} \sum_{v \in V} \alpha(v) \beta_v(B)$

$$\mathbb{P}_{D_0}(v_\tau = v) \longrightarrow \frac{1}{|S(G)|} \sum_{B \in S(G)} \alpha(v)\, \beta_v(B)$$

$$= \alpha(v)$$

$$\mathbb{P}_{D_0}(e_\tau = e \mid v_\tau = v, B_\tau = B) \to \begin{cases} \frac{1}{\beta_v(B)} & \text{if } 0 \le e < \beta_v(B) \\ 0 & \text{o.w.} \end{cases}$$

$$\mathbb{P}_{D_0}(\deg(D_\tau) = n) \longrightarrow \frac{|\mathcal{B}^n|}{|S(G)|}$$

$$\mathcal{B}^n := \{B \in \mathcal{B}(G,s) : \deg(B) = n\}$$

Pf: $D_\tau = B_\tau - L\sigma$

$\Rightarrow \deg(D_\tau) = \deg(B_\tau)$

$$\mathbb{P}_{D_0}(\deg(D_\tau) = n) = \sum_{v \in V} \mathbb{P}_{D_0}(v_\tau = v, \deg(D_\tau) = n)$$

$$D_\tau = B_\tau^v - L\sigma_\tau^v$$

$$\Rightarrow \deg(D_\tau) = \deg(B_\tau^v)$$

$$\mathbb{P}_{D_0}(v_\tau = v, \deg(D_\tau) = n) = \mathbb{P}_{D_0}(v_\tau = v, \deg(B_\tau^v) = n)$$

$$= \sum_{B \in \mathcal{B}^n} \mathbb{P}_{D_0}(v_\tau = v, B_\tau^v = B)$$

v is sink so $\beta_v(B_\tau^v) = 1$

$$\to \sum_{B \in \mathcal{B}^n} \frac{\alpha(v)}{|S(G)|}$$

$$= \frac{|\mathcal{B}^n|\,\alpha(v)}{|S(G)|}$$

$$\mathbb{P}_{D_0}(\deg(D_\tau)=n) \longrightarrow \frac{|\mathcal{B}^n|}{|S(G)|}$$

$\square$

Defn: $\zeta_\tau(D_0) := \dfrac{\mathbb{E}_{D_0} \deg(D_\tau)}{|V|}$

$$\zeta_{st} := \frac{1}{|S(G)|} \sum_{B \in B(G)} \frac{\deg(B)}{|V|}$$

Threshold density theorem

$G=(V,E)$ Eulerian

$$\zeta_\tau(D_0) \longrightarrow \zeta_{st} \quad \text{as} \quad \deg(D_0) \longrightarrow -\infty$$

Pf:

$$X \sim \text{Uniform}(B(G,S))$$

$$\therefore \quad \deg(D_\tau) \xrightarrow{d} \deg(X)$$

$$|\deg(D_\tau)| \leq \sum_v \deg(v)$$

$$= 2|E|$$

$\therefore$ By BCT,

$$\underset{\substack{\| \\ \zeta_\tau(D_0)\cdot|V|}}{\mathbb{E}_{D_0}[\deg(D_\tau)]} \longrightarrow \underset{\substack{\| \\ \zeta_{st}\cdot|V|}}{\mathbb{E}[\deg(X)]}$$

$\square$

Time Permitting:

Thm (Merino)

$$T(1,y) = \sum_{i=0}^{g} h_{g-i}\, y^i$$

h_{g-i} ↓ no. of superstables of deg $g-i$

Defn: $G=(V,E)$ undirected connected multigraph

$$t_G(y) := T(G;1,y)$$

Theorem: $$S_{s.t.}(G) = \frac{1}{|V|}\left(|E| + \frac{t_G'(1)}{t_G(1)}\right)$$

Pf:

$$t_G(y) = T(1,y) = \sum_{i=0}^{g} h_{g-i}\, y^i$$

$$t_G'(1) = \sum_{i=0}^{g} i\, h_{g-i}$$

$$\sum_{c\in S(G)} \deg(c_{max}-c) = \sum_{i=0}^{g} (g-i) h_{g-i}$$

$$= g\, t_G(1) - t_G'(1)$$

$$= g|S(G)| - t_G'(1)$$

$$\Rightarrow \deg(c_{max}) - \frac{1}{S(G)} \sum_{c\in S(G)} \deg(c) = g - \frac{t_G'(1)}{t_G(1)}$$

$$\deg(c_{max}) = \sum_{v\in V} (\deg(v) - 1) - \deg_G(s) + 1$$

$$= 2|E| - |V| + 1 - \deg_G(s)$$

$$= |E| + g - \deg_G(s)$$

$$|E| + g - \deg_G(s) - \frac{1}{S(G)} \sum_{c \in S(G)} \deg(c) = g - \frac{t_G'(1)}{t_G(1)}$$

$$\Rightarrow -|E| + \deg_G(s) + \frac{1}{S(G)} \sum_{c \in S(G)} \deg(c) = \frac{t_G'(1)}{t_G(1)}$$

$$\Rightarrow \frac{\deg_G(s)}{|V|} + \frac{1}{S(G)} \sum_{c \in S(G)} \frac{\deg(c)}{|V|} = \frac{1}{|V|}\left(|E| + \frac{t_G'(1)}{t_G(1)}\right)$$

$$\Rightarrow \varsigma_{st} = \frac{1}{|V|}\left(|E| + \frac{t_G'(1)}{t_G(1)}\right)$$

$\square$

Theorem: $t_G'(1) = \#$ spanning unicycles

$\downarrow$

an edge + tree

Ramanujan's Q function: $Q(n) = \sum_{k=1}^{n} \frac{n^{\underline{k}}}{n^k}$ $\quad n^{\underline{k}} := n(n-1)\cdots(n-k+1)$

Thm: $\varsigma_{st}(K_n) = \frac{1}{2}\left(Q(n) + n - 3 + \frac{1}{n}\right)$

Pf: $\#$ spanning unicycles

$$= \sum_{k=3}^{n} \underbrace{k\, n^{n-k-1}}_{\text{no. of } k\text{-rooted forests}} \underbrace{\binom{n}{k}}_{\text{choose } k \text{ roots}} \underbrace{\frac{k!}{2k}}_{\text{no. of ways to arrange } k\text{-beads in a cycle}}$$

$$= \frac{n^{n-1}}{2} \sum_{k=3}^{n} \frac{n^{\underline{k}}}{n^k}$$

$$Q(n) = \sum_{k \geq 1} \frac{n^{\underline{k}}}{n^k} = 1 + \frac{n-1}{n} + \sum_{k=3}^{n} \frac{n^{\underline{k}}}{n^k}$$

$$= 2 - \frac{1}{n} + \frac{2t_G'(1)}{n^{n-1}}$$

$$\Rightarrow t_G'(1) = \frac{1}{2} n^{n-2} \left(nQ(n) - 2n + 1 \right)$$

$$\therefore \zeta_{st.}(K_n) = \frac{1}{n} \left(\frac{n(n-1)}{2} + \frac{1}{2} \left(nQ(n) - 2n + 1 \right) \right)$$

$$= \frac{n-1}{2} + \frac{1}{2} Q(n) - 1 + \frac{1}{2n}$$

$$= \frac{1}{2} \left(Q(n) + n - 3 + \frac{1}{n} \right)$$

$\square$

Remark: $Q(n) \sim \sqrt{\frac{\pi n}{2}}$

$$\Rightarrow \zeta_{s.t.}(K_n) \sim \frac{n}{2}$$