Setting

X : Hausdorff Topological space

Riemann Surfaces

↳ Surfaces i.e. 2 dimensional manifold i.e. locally like $\mathbb{R}^2$ or $\mathbb{C}$.

↳ in the intersection of these ngbd is conformally same as an open subset of $\mathbb{C}$

↳ why? [Because this allows to lift notions of holomorphicity to general spaces]

Chart: $\varphi : U \to V$ is a homeo from $U \overset{open}{\subset} X$ to $V \overset{open}{\subset} \mathbb{C}$.

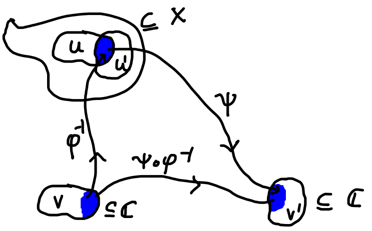
Notational Convention

$$\varphi : U \to V \qquad \varphi_0 : U \cap U' \longrightarrow \varphi(U \cap U') \qquad \varphi_0 = \varphi|_{U\cap U'}$$

$$\psi : U' \to V' \qquad \psi_0 : U \cap U' \longrightarrow \psi(U \cap U') \qquad \psi_0 = \psi|_{U \cap U'}$$

$$\psi \circ \varphi^{-1} := \psi_0 \circ \varphi_0^{-1}$$

↳ chart transformation.



Atlas: set of charts $\varphi : U_\varphi \to V_\varphi$ s.t. $X = \bigcup_{\varphi \in \mathcal{A}} U_\varphi$

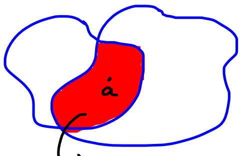
now let us come to the analytic part.

Analytically compatible: φ & ψ are Analytically compatible if $\psi \circ \varphi^{-1}$ is conformal.

Analytic: $\mathcal{A}$ on X is called analytic if every pair φ & ψ is conformal.

Now, take $\mathcal{A}$ & $\mathcal{B}$ s.t. $\mathcal{A} \cup \mathcal{B}$ analytic

then



↳ $\varphi_{\mathcal{A}} \circ \varphi_{\mathcal{B}}^{-1}$ is conformal

essentially

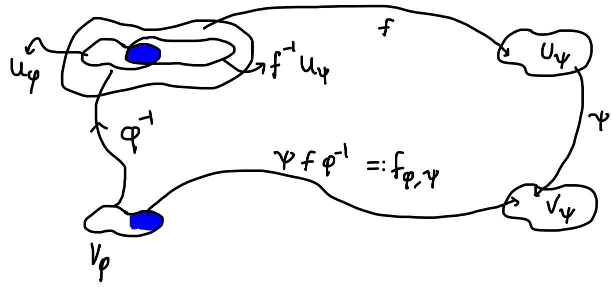
$\varphi_{\mathcal{A}}$ & $\varphi_{\mathcal{B}}$ are same locally i.e. $\varphi_{\mathcal{A}}$ & $\varphi_{\mathcal{B}}$ are analytically compatible

so such atlases will be caled essentially equal.

Riemann Surfaces.

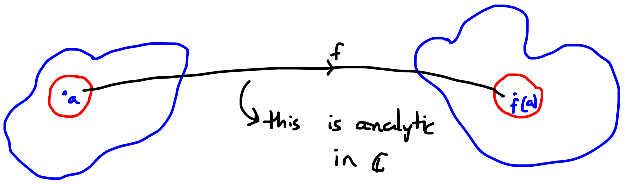
$(X, [\mathcal{A}])$ consists of topological space & a class of analytic atlases.

Now let us define analytic maps.



A cont. map $f: X \to Y$ is called analytic at $a \in X$ if $\exists\ \varphi \in \mathcal{A}$ & $\psi \in \mathcal{B}$ s.t. $a \in U_\varphi$, $f(a) \in U_\psi$ & $f_{\varphi,\psi}$ is analytic at $\varphi(a)$.

Intuition



But in order to look into this ngbds of a & $f(a)$ you need two telescopes φ^{-1} & ψ.

(i) $id_X : X \longrightarrow X$ analytic

(ii) $X \xrightarrow{f} Y \xrightarrow{g} Z$ $\quad g \circ f$ is analytic

$\therefore$ Riemann Surfaces form a category with analytic maps as the morphisms.

Equivalence:

$X \underset{g}{\overset{f}{\rightleftarrows}} Y$ $\quad f \circ g = 1_Y$ & $g \circ f = 1_X$ & f, g are analytic

f is called biholomorphic or conformal.

Subsurfaces of R.S.

$U \overset{open}{\subset} X$ $\quad (U, \mathcal{A}|U)$

$\mathcal{A}|U := \{\varphi|_U \mid \varphi \in \mathcal{A}\}$

Eg. $(\mathbb{C}, \{id\})$ is Riemann Surface.

Once you have an example

We can have the functor $Hom(-, \mathbb{C})$

$\hookrightarrow$ i.e. analytic maps from $X \to \mathbb{C}$

they are called analytic functions

$Hom(-, \mathbb{C})$ and why not $Hom(\mathbb{C}, -)$

$\because$ $Hom(-, \mathbb{C})$ this functors map it RS $\to$ Rings why?

$\mathcal{O}(X) := Hom(X, \mathbb{C})$ set of analytic fns.

- $f, g \in \mathcal{O}(X)$
- $f+g \in \mathcal{O}(X)$ $\quad f.g \in \mathcal{O}(X)$
- $cf \in \mathcal{O}(X)$ $\quad c \in \mathbb{C}$

Forget functor think it as a ring

More questions,

Is $\mathcal{O}(X)$ integral domain?

What about units?

$\mathcal{O}(\mathbb{C})$ is integral domain

$f(x)\cdot g(x) = 0 \quad f \not\equiv 0$

$\Rightarrow g(x) = 0 \ \forall\ x \in \mathbb{C} \setminus X$ ↗ discrete set $X = f^{-1}(0)$

$\Rightarrow g \equiv 0$ by identity theorem

↳ if you want to prove ID then you need identity theorem.

Eg 2. Riemann Sphere : $[\bar{\mathbb{C}}, \mathcal{A}]$

$$\left.\begin{array}{l} \bar{\mathbb{C}} \setminus \{\infty\} \xrightarrow{id_{\mathbb{C}}} \mathbb{C} \\ \quad z \longmapsto z \\ \bar{\mathbb{C}} \setminus \{0\} \longrightarrow \mathbb{C} \\ \quad z \longmapsto 1/z \end{array}\right\} \mathcal{A}$$

chart transformation : $\mathbb{C} \setminus \{0\} \longrightarrow \mathbb{C} \setminus \{0\}$ conformal

$z \rightarrow \frac{1}{z}$

$\mathrm{Hom}(X, \bar{\mathbb{C}})$: All the analytic functions from X to $\bar{\mathbb{C}}$.

$f : X \longrightarrow \bar{\mathbb{C}}$ s.t. $f^{-1}(\infty)$ is a discrete subset of X. → meromorphic functions

$f : \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$ ⟵ meromorphic in complex analysis sense.

$\mathcal{O}(X) \hookrightarrow \mathcal{M}(X)$

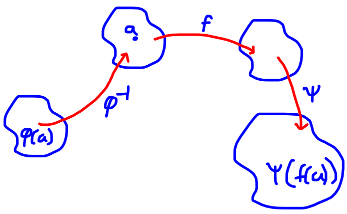
↳ this is indeed a field in some cases

↳ this will also prove $\mathcal{O}(X)$ is integral domain.

Thm: X conn $S \subseteq X$ where S has limit pt. & $f(s) = 0 \Rightarrow f \equiv g$ in X

Pf: X is locally path conn.

$\Rightarrow X$ is path conn.



Let $a \in S$ be a limit pt.

$\mathcal{U} \ni a$ s.t. $\mathcal{U}$ is conn.

inside co-ordinate ngbd

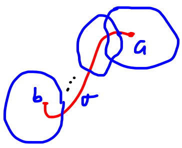
$f(\mathcal{U})$ inside co-ordinate ngbd

$\Rightarrow \varphi(S \cap \mathcal{U})$ has limit p.t.

& $\psi \circ f \circ \varphi^{-1} = \psi \circ g \circ \varphi^{-1}$ on $\varphi(S \cap \mathcal{U})$

$\Rightarrow \psi \circ f \circ \varphi^{-1} = \psi \circ g \circ \varphi^{-1}$ on $S \cap \mathcal{U}$

$\Rightarrow f = g$ on $S \cap \mathcal{U}$



Let $b \in X$ $\exists$ path σ.

Cover σ by $\mathcal{U}_1, \dots, \mathcal{U}_n$ s.t. $\mathcal{U}_i$ are nice ngbds. numbers them in such a way that $\mathcal{U}_i \cap \mathcal{U}_{i+1} \neq \phi$.

$\mathcal{U}_i \cap \mathcal{U}_{i+1}$ is homeomorphic to infinite set

hence has limit pt.

by induction $f = g$ on $\mathcal{U}_n$

$\Rightarrow f(b) = g(b)$

$\Rightarrow f = g$ on X.

Thm: $\mathcal{M}(X)$ is a field.

Pf: Let $f \neq 0$.

$g = \frac{1}{f}$ is meromorphic & $g(Z(f)) = \infty$

$\Rightarrow g$ is analytic at $X \setminus Z(f)$

& $Z(f)$ is discrete.

$\because \frac{1}{g(\varphi^{-1}(z))}$ is analytic at $Z(f)$

$\Rightarrow g$ is analytic $X \to \overline{\mathbb{C}}$ & $g^{-1}(\infty) = Z(f)$ is discrete.

$\Rightarrow g \in \mathcal{M}(X)$

$\therefore \mathcal{M}(X)$ is a field & $\mathcal{O}(X)$ is i.d.

$\mathcal{M}: RS \longrightarrow$ Fields

Is $Frac(\mathcal{O}(X)) = \mathcal{M}(X)$?

No (Not always)

Thm: X - cpt connected R.S. $\Rightarrow$ the only analytic functions are constant functions.

Pf: Let $U \overset{op}{\subseteq} X$

$\exists$ a s.t. f attains maximum at a.

Let $a \in U_\varphi$

$\Rightarrow f \circ \varphi^{-1} : V_\varphi \longrightarrow \mathbb{C}$ attains maxima at a

$\Rightarrow f \circ \varphi^{-1}$ is constant

$\Rightarrow f$ is const

By prev. thm f is const everywhere.

This shows $\mathcal{O}(X)$ only consists of constant functions whereas $\mathcal{M}(X)$ can consist of non-constant functions

Thm: X - connected R.S. $f: X \longrightarrow Y$ is an non-const. holomorphic map $\Rightarrow f$ is open.

Pf: We will prove $f(a) \in f(X)^\circ$

take $U_\varphi \ni a$ & $U_\psi \ni f(a)$ s.t. U_φ is conn. & $f(U_\varphi) \subseteq U_\psi$.

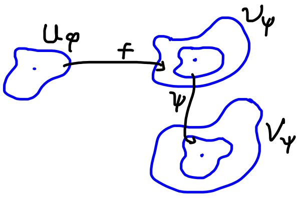
$\psi \circ f \circ \varphi^{-1}$ is an open map $V_\varphi \longrightarrow V_\psi$

$\therefore \psi(f(a)) \in f_{\varphi,\psi}(V_\varphi)$

$f(a) \in \psi^{-1}(f_{\varphi,\psi}(V_\varphi))$

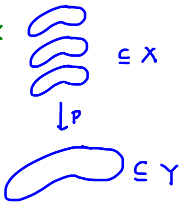
& V_ψ is open $\Rightarrow \psi^{-1}(V_\psi)$ is open.

$\Rightarrow f(a) \in f(X)^\circ$



Covering Spaces

Thm: Y : R.S. $p : X \to Y$ covering. $\exists !$ Riemann surface structure on X w.r.t. which p is holomorphic.



Pf. Existence

let $x \in X$ take $\mathcal{U}_\varphi \ni p(x)$ evenly covered ngbd of $p(x)$
& $\mathcal{U}_\varphi$ is also a co-ordinate ngbd

take $U_x \ni x$ s.t. $p|_{U_x} : U_x \to \mathcal{U}$ is homeomorphism

Now, $U_x \xrightarrow{p} \mathcal{U}_\varphi \xrightarrow{\varphi} V_\varphi$

$\therefore$ $\varphi \circ p$ is a homeo. on U_x co-ordinate ngbd of x.

$\therefore$ $\mathcal{A}_x = \{\varphi \circ p\}_{\varphi \in \mathcal{A}_Y}$

& $\psi \circ p \circ p^{-1} \circ \varphi^{-1} = \psi \circ \varphi^{-1} \to$ conformal

$\psi \circ p \circ p^{-1} \circ \varphi^{-1} = \psi \circ \varphi^{-1} \Rightarrow p$ is analytic

Uniqueness

Lemma: X_1, X_2 coverings of Y. Diagram commutes then f is holomorphic

$$\begin{tikzcd} & X_2 \arrow[d, "p_2"] \\ X_1 \arrow[ur, "f"] \arrow[r, "p_1"'] \arrow[ur, phantom, shift right=3ex, "\circlearrowright"] & Y \end{tikzcd}$$

Pf: $x_1 \in X_1$ $y = p_1(x_1) = p_2(x_2)$ where $x_2 = f(x_1)$

take $\mathcal{U} \ni y$ co-ordinate ngbd.

$U_1 \ni x_1$ $\mathcal{U}_2 \ni x_2$ s.t. $p|_{U_i} U_i \xrightarrow{\sim} \mathcal{U}$.

$\psi \circ p_2 \circ f \circ (\varphi \circ p_1)^{-1}$

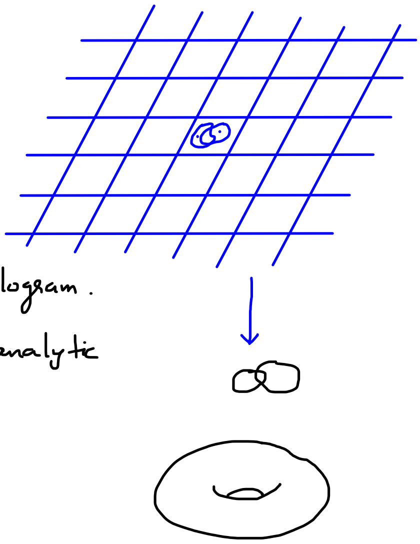
$= \psi \circ p_1 \circ p_1^{-1} \circ \varphi^{-1}$

$= \psi \circ \varphi^{-1} \to$ analytic

This proves uniqueness [$\because$ isomorphic topologically $\Rightarrow$ isomorphic conformally]

Lattice: $\omega_1 \mathbb{Z} + \omega_2 \mathbb{Z} \subseteq \mathbb{C}$

$\mathbb{C}/\Gamma$ is a Riemann surface



take $a \in \mathbb{C}/\Gamma$

$\mathcal{U} \ni a$ evenly covered

map it to the fundamental parallelogram.

$\psi \circ \varphi^{-1}$ is translation and hence analytic

$\Rightarrow S^1 \times S^1$ is Riemann Surface