Limit Theorems of estimators in the Tensor Curie Weiss Potts Model

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(joint work with S. Mukherjee[2])

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2 Goal

3 Asymptotics of Magnetization vector

4 Asymptotics of the Maximum Likelihood Estimates

Spin Glass Model

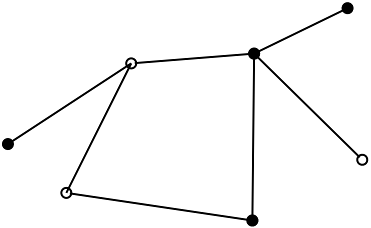

Figure: Magnetic Spins in a graph with black vertices denoting -1 and white vertices denoting $+1$

$\sigma \in \{-1, 1\}^n$ denotes the magnetic spin vector in the graph.

$$\mathbb{P}(\sigma) \propto \exp(\beta \sum_{j=1}^{n} J_{ij}\sigma_i\sigma_j)$$

where J: interaction matrix, $\beta > 0$: inverse temperature.

Potts Model

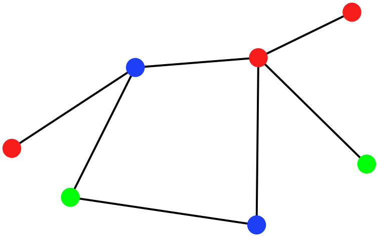

Figure: Coloring with 3 colors in a graph(may not be proper)

$X \in [q]^N$ denotes the coloring in the graph.

$$\mathbb{P}(X) \propto \exp\left(\beta \sum_{j=1}^{n} J_{ij} \mathbb{1}_{X_i = X_j}\right)$$

Tensor Potts Model

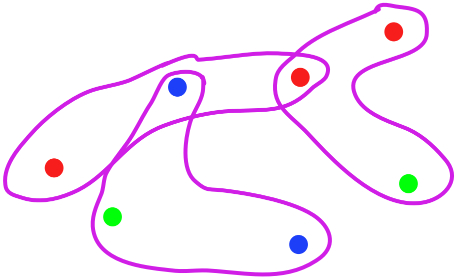

Figure: Introducing peer group interactions

$X \in [q]^n$ denotes the coloring in the hypergraph.

$$\mathbb{P}(X) \propto \exp\left(\beta \sum_{i_1,\ldots,i_p} J_{i_1,\ldots,i_p} \mathbb{1}_{X_{i_1}=\cdots=X_{i_p}} + h \sum_{i=1}^{N} 1_{X_i=1}\right)$$

where, $h \geq 0$: magnetic field.

Set Up

- Consider the Curie-Weiss setting where $J_{i_1,\ldots,i_p} = \frac{1}{N^{p-1}}$.

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- Define $\bar{X}_r = \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{X_i = r}$ then,

$$\mathbb{P}_{\beta,h,N}(\boldsymbol{X}) := \frac{1}{q^N Z_N(\beta, h)} \exp\left(\beta N \sum_{r=1}^{q} \bar{X}_r^p + N h \bar{X}_{.1}\right) \quad (1.1)$$

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- Magnetization Vector, $\bar{\boldsymbol{X}}_N := (\bar{X}_r)_{r=1}^{q}$

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Goal

- Asymptotics of the maximum likelihood (ML) estimates of the parameters β and h.
- Asymptotics of the magnetization vector $(\bar{\boldsymbol{X}}_N)$.

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Large Deviation of $\bar{\boldsymbol{X}}_N$

$\bar{\boldsymbol{X}}_N$ concentrates around global maximizers of $H_{\beta,h}$:

$$H_{\beta,h}(\boldsymbol{t}) := \beta \sum_{r=1}^{q} t_r^p + ht_1 - \sum_{r=1}^{q} t_r \log t_r$$

Theorem (B.,Mukherjee (2023))

Let $\beta_N \to \beta$ and $h_N \to h$. Then, under $\mathbb{P}_{\beta_N,h_N,N}$, the empirical magnetization $\bar{\boldsymbol{X}}_N$ satisfies a large deviation principle with speed N and rate function $-H_{\beta,h} + \sup H_{\beta,h}$.

Partitioning the Parameter Space

1. Regular: if the function $H_{\beta,h}$ has a unique global maximizer $\boldsymbol{m}_*$ and the $\boldsymbol{Q}_{\boldsymbol{m}_*,\beta} = Hess(H_{\beta,h})$ is negative definite at $\boldsymbol{m}_*$ on $\mathcal{H}_q := \{\boldsymbol{t} \in \mathbb{R}^q : \sum_{r=1}^{q} t_r = 0\}$.

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2. Critical: if $H_{\beta,h}$ has more than one global maximizer.
3. Special: if $H_{\beta,h}$ has a unique global maximizer $\boldsymbol{m}_*$ and $\boldsymbol{Q}_{\boldsymbol{m}_*,\beta}$ is singular on $\mathcal{H}_q$.

Form of the maximizers

Lemma

The maximizers $\boldsymbol{m}$ are of the form $\left(\frac{1+(q-1)s}{q}, \frac{1-s}{q}, \cdots, \frac{1-s}{q}\right)$

$$f_{\beta,h}(s) := (q-1)k\left(\frac{1-s}{q}\right) + k\left(\frac{1+(q-1)s}{q}\right) + \left(\frac{1+(q-1)s}{q}\right) \cdot h,$$

where $k(x) = k_{\beta,p}(x) := \beta x^p - x \log x$.

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where $k(x) = k_{\beta,p}(x) := \beta x^p - x \log x$.
Special:

1. type-I, if $f^{(4)}_{\beta,h}(s) < 0$.
2. type-II, if $f^{(4)}_{\beta,h}(s) = 0$(Not observed in Ising Model)

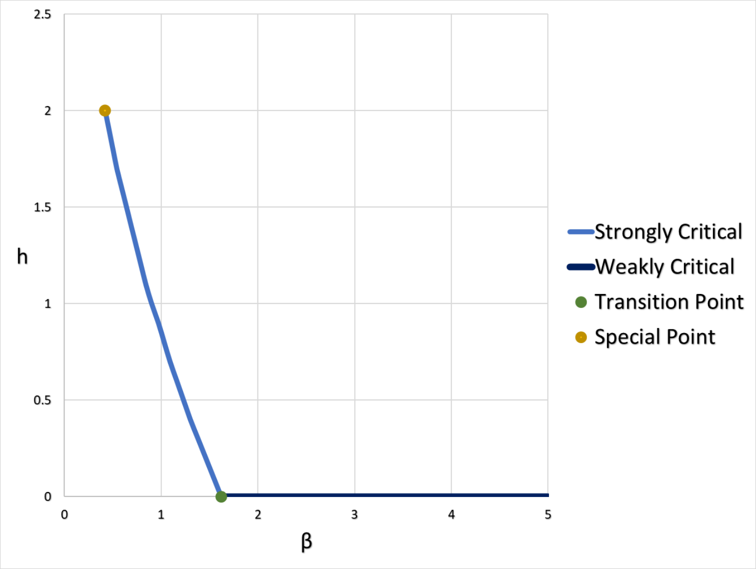

Figure: Partition of the paramter space, for p=7,q=5

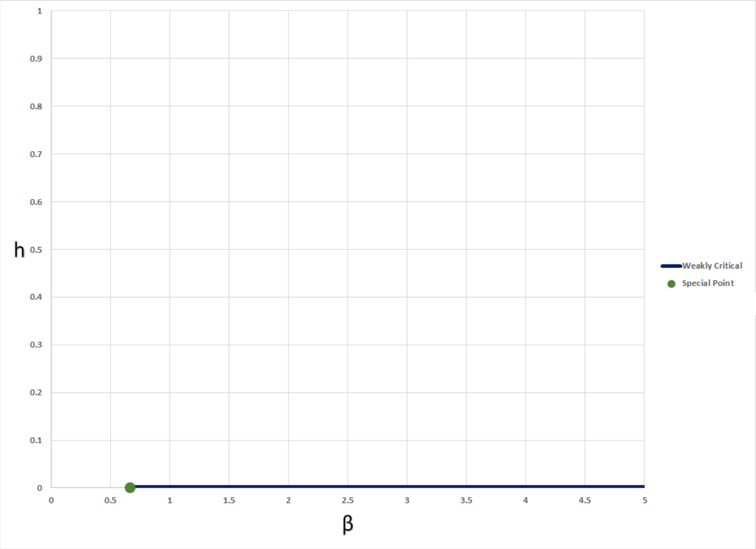

Figure: Partition of the paramter space, for p=4,q=2

CLT of $\bar{\boldsymbol{X}}_N$ under perturbation

Theorem (B.,Mukherjee(2023))

1. *Regular: for* $\boldsymbol{X} \sim \mathbb{P}_{\beta+N^{-\frac{1}{2}}\bar{\beta},h+N^{-\frac{1}{2}}\bar{h},p}$ *for some* $\bar{\beta}, \bar{h} \in \mathbb{R}$,

$$N^{\frac{1}{2}}\left(\bar{\boldsymbol{X}}_N - \boldsymbol{m}_*\right) \xrightarrow{D} \mathcal{N}_q\left(\Sigma(\bar{\beta}p\boldsymbol{m}_*^{p-1} + \bar{h}\boldsymbol{e}_1), \Sigma\right),$$

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2. *Critical:*
   - *for* $\boldsymbol{X} \sim \mathbb{P}_{\beta,h,p}$, *as* $N \to \infty$, *we have:*

$$\bar{\boldsymbol{X}}_N \xrightarrow{P} \sum_{k=1}^{K} p_k \delta_{m_k},$$

   - *under* $\mathbb{P}_{\beta_N,h_N}\left(\cdot \middle| \bar{\boldsymbol{X}}_N \in B(\boldsymbol{m}_i,\varepsilon)\right)$:

$$\sqrt{N}\left(\bar{\boldsymbol{X}}_N - \boldsymbol{m}_i\right) \xrightarrow{D} \mathcal{N}_q(\Sigma'(\bar{\beta}p\boldsymbol{m}_i^{p-1} + \bar{h}\boldsymbol{e}_1), \Sigma'),$$

CLT of $\bar{\boldsymbol{X}}_N$ under perturbation

Theorem ((contd.))

3. *Special: Let* $\boldsymbol{u} =:= (1-q, 1, \ldots, 1)$,
   - *Type-I: Under* $\mathbb{P}_{\beta+N^{-\frac{3}{4}}\bar{\beta}, h+N^{-\frac{3}{4}}\bar{h}, p}$, *as* $N \to \infty$,

$$N^{\frac{1}{4}}(\bar{\boldsymbol{X}}_N - \boldsymbol{m}_*) \xrightarrow{D} T_{\bar{\beta},\bar{h}}\boldsymbol{u},$$

where $T_{\bar{\beta},\bar{h}}$ *has density proportional to,*

$$\exp\left(\frac{x^4}{24}q^4 f^{(4)}_{\beta,h}(s) + (\bar{\beta}p\langle \boldsymbol{m}_*^{p-1}, \boldsymbol{u}\rangle + \bar{h}(1-q))x\right).$$

CLT of $\bar{X}_N$ under perturbation

Theorem ((contd.))

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   - *Type-II: Under* $\mathbb{P}_{\beta+N^{-\frac{5}{6}}\bar{\beta}, h+N^{-\frac{5}{6}}\bar{h}, p}$,

     $$N^{\frac{1}{6}}\left(\bar{\boldsymbol{X}}_N - \boldsymbol{m}_*\right) \xrightarrow{D} F_{\bar{h}}\boldsymbol{u}.$$

     where $F_{\bar{h}}$ *has density proportional to* $\exp\left(-\frac{32}{15}x^6 - \bar{h}x\right)$.

Proof Sketch

- Denote $\boldsymbol{W}_N := \sqrt{N}\left(\bar{\boldsymbol{X}}_N - \boldsymbol{m}_*\right)$.
- $g : \mathbb{R}^q \to \mathbb{R}$ a bounded, continuous function .

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- Weak Convergence:

$$q^N Z_N(\beta_N, h_N)\mathbb{E}_{\beta_N,h_N,N}\left[g(\boldsymbol{W}_N)\mathbb{1}_{\|\boldsymbol{W}_N\|\le M}\right]$$
$$= \sum g(\boldsymbol{w}(\boldsymbol{v}))\mathbb{1}_{\|\boldsymbol{w}(\boldsymbol{v})\|\le M} q^N Z_N(\beta_N, h_N)\mathbb{P}_{\beta_N,h_N,N}(\bar{\boldsymbol{X}}_N = \boldsymbol{v})$$

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- A small lemma,

$$q^N Z_N(\beta, h)\mathbb{P}_{\beta, h, N}(\bar{\boldsymbol{X}}_N = \boldsymbol{v}) \sim N^{-\frac{q-1}{2}} A(\boldsymbol{v}) e^{N H_{\beta, h}(\boldsymbol{v})}$$

where $A(\boldsymbol{v}) := (2\pi)^{-(q-1)/2} \prod_{r=1}^{q} v_r^{-1/2}$.

Proof Sketch

Taylor expand H_{β_N,h_N} to get,

$$q^N Z_N(\beta_N, h_N)\mathbb{E}_{\beta_N,h_N,N}\left[g(\boldsymbol{W}_N)\mathbb{1}_{\|\boldsymbol{W}_N\|\leq M}\right]$$

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$$\sim \frac{A(\boldsymbol{m}_*)}{N^{\frac{q-1}{2}}} e^{NH_{\beta_N,h_N}(\boldsymbol{m}_*)} \sum g(\boldsymbol{w})\mathbb{1}_{\|\boldsymbol{w}\|\le M} e^{\langle \bar{\beta} p \boldsymbol{m}_*^{p-1} + \bar{h}\boldsymbol{e}_1, \boldsymbol{w}\rangle + \frac{1}{2}\boldsymbol{Q}_{\boldsymbol{m}_*,\beta}(\boldsymbol{w})}$$

Proof Sketch

Taylor expand H_{β_N,h_N} to get,

$$\begin{aligned}
&q^N Z_N(\beta_N, h_N)\mathbb{E}_{\beta_N,h_N,N}\left[g(\boldsymbol{W}_N)\mathbb{1}_{\|\boldsymbol{W}_N\|\le M}\right] \\
\sim&\frac{A(\boldsymbol{m}_*)}{N^{\frac{q-1}{2}}}e^{NH_{\beta_N,h_N}(\boldsymbol{m}_*)}\sum g(\boldsymbol{w})\mathbb{1}_{\|\boldsymbol{w}\|\le M}e^{\langle\bar{\beta}p\boldsymbol{m}_*^{p-1}+\bar{h}\boldsymbol{e}_1,\boldsymbol{w}\rangle+\frac{1}{2}\boldsymbol{Q}_{\boldsymbol{m}_*,\beta}(\boldsymbol{w})} \\
\sim&A(\boldsymbol{m}_*)e^{NH_{\beta_N,h_N}(\boldsymbol{m}_*)}\int_{\mathcal{H}_q\bigcap B(\boldsymbol{0},M)}g(\boldsymbol{w})e^{\langle\bar{\beta}p\boldsymbol{m}_*^{p-1}+\bar{h}\boldsymbol{e}_1,\boldsymbol{w}\rangle+\frac{1}{2}\boldsymbol{Q}_{\boldsymbol{m}_*,\beta}(\boldsymbol{w})}.
\end{aligned}$$

Proof Sketch

Hence, under $\mathbb{P}_{\beta_N,h_N,N}$, $\boldsymbol{W}_N$ conditioned on $\|\boldsymbol{W}_N\| \leq M$ converges weakly to the density proportional to

$$\boldsymbol{w} \mapsto e^{\langle \bar{\beta} p \boldsymbol{m}_*^{p-1} + \bar{h} \boldsymbol{e}_1, \boldsymbol{w} \rangle + \frac{1}{2} \boldsymbol{Q}_{\boldsymbol{m}_*,\beta}(\boldsymbol{w})} .$$

Conclude the proof by uniform integrability of $\{X_N\}$.

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MLE of the parameters

We interested to esimate:

1. β with h known,
2. h with β known.

MLE of the parameters

Define $u_{N,p}$ and $u_{N,1}$ as,

$$u_{N,p}(\beta, h, p) := \mathbb{E}_{\beta,h,p}(\|\bar{\boldsymbol{X}}_N\|_p^p) \quad \text{and} \quad u_{N,1}(\beta, h, p) := \mathbb{E}_{\beta,h,p}(\bar{X}_1) .$$

From classical results of exponential family the ML estimate $\hat{\beta}$ satisfies the equation for fixed h:

$$u_{N,p}(\beta, h, p) = \|\bar{\boldsymbol{X}}_N\|_p^p$$

and for fixed β, the ML estimate $\hat{h}$ satisfies the equation:

$$u_{N,1}(\beta, h, p) = \bar{X}_1.$$

A Lemma

Lemma

$u_{N,p}(\beta, h)$ and $u_{N,1}(\beta, h)$ are increasing in β and h respectively.

Proof Sketch:
Define $F_N := \log q^N Z_N(\beta, h)$. Then note that $\frac{\partial}{\partial \beta} F_N = u_{N,p}(\beta, h)$ and $\frac{\partial}{\partial h} F_N = u_{N,1}(\beta, h)$. From holder's inequality F_N is convex in β, h.

Limit Theorems of MLE

Theorem (B., Mukherjee(2023))

For regular points we have:

$$N^{\frac{1}{2}}\left(\hat{h}_N - h\right) \xrightarrow{D} \mathcal{N}\left(0, -\frac{q^2}{(q-1)^2} f''_{\beta,h}(s)\right)$$

Proof.

$$\mathbb{P}_{\beta,h,p}\left(N^{\frac{1}{2}}\left(\hat{h}_N - h\right) \leq t\right)$$

$$= \mathbb{P}_{\beta,h,p}\left(\hat{h}_N \leq h + \frac{t}{N^{\frac{1}{2}}}\right)$$

$$= \mathbb{P}_{\beta,h,p}\left(u_{N,1}\left(\beta, \hat{h}_N, p\right) \leq u_{N,1}\left(\beta, h + \frac{t}{N^{\frac{1}{2}}}, p\right)\right)$$

Limit Theorems of MLE

Proof(contd.)

$$
\begin{aligned}
&= \mathbb{P}_{\beta,h,p}\left(\bar{X}_1 \le \mathbb{E}_{\beta,h+N^{-\frac{1}{2}}t,p}\left(\bar{X}_1\right)\right) \\
&= \mathbb{P}_{\beta,h,p}\left(N^{\frac{1}{2}}\left(\bar{X}_1 - m_1\right) \le \mathbb{E}_{\beta,h+N^{-\frac{1}{2}}t,p}\left(N^{\frac{1}{2}}\left(\bar{X}_1 - m_1\right)\right)\right) \\
&\to \mathbb{P}_{\beta,h,p}\left(\mathcal{N}\left(0, -\frac{(q-1)^2}{q^2 f''_{\beta,h}(s)}\right) \le -\frac{t(q-1)^2}{q^2 f''_{\beta,h}(s)}\right) \\
&= \mathbb{P}_{\beta,h,p}\left(\mathcal{N}\left(0, -\frac{q^2 f''_{\beta,h}(s)}{(q-1)^2}\right) \le t\right).
\end{aligned}
$$

□

Asymptotics of $\hat{\beta}_N$ and $\hat{h}_N$

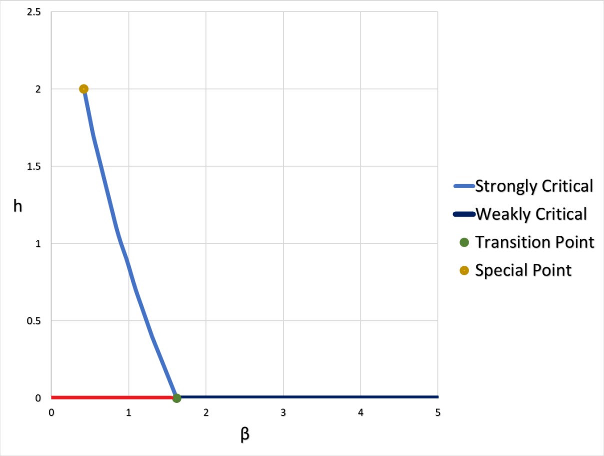

Figure: Regular: $\hat{\beta}_N$ is $\sqrt{N}$ consistent except at the red line. $\hat{h}_N$ is $\sqrt{N}$ consistent

Asymptotics of $\hat{\beta}_N$ and $\hat{h}_N$

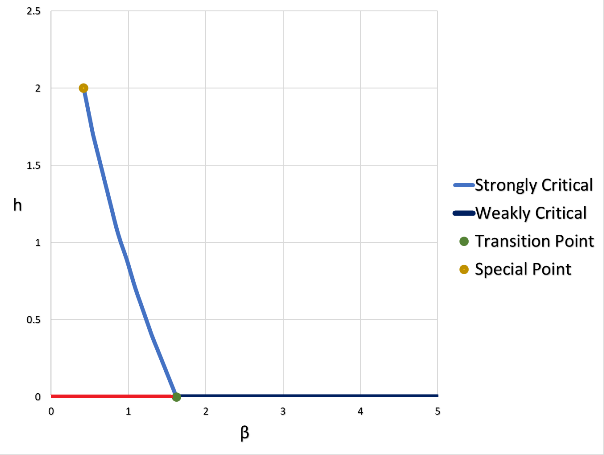

Figure: Strongly Critical: $\hat{\beta}_N$ is $\sqrt{N}$ consistent. $\hat{h}_N$ is $\sqrt{N}$ consistent

Asymptotics of $\hat{\beta}_N$ and $\hat{h}_N$

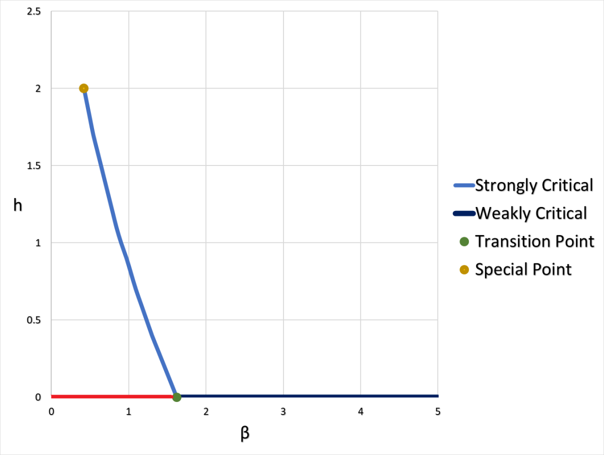

Figure: Weakly Critical: $\hat{\beta}_N$ is $\sqrt{N}$ inconsistent. $\hat{h}_N$ is $\sqrt{N}$ consistent

Asymptotics of $\hat{\beta}_N$ and $\hat{h}_N$

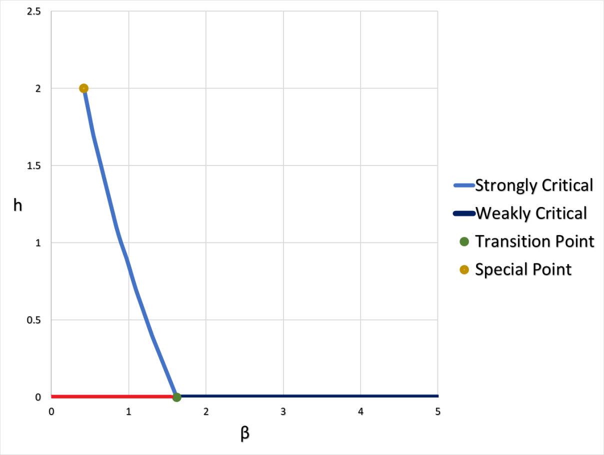

Figure: Transition Point: $\hat{\beta}_N$ is $\sqrt{N}$ inconsistent. $\hat{h}_N$ is $\sqrt{N}$ consistent

Asymptotics of $\hat{\beta}_N$ and $\hat{h}_N$

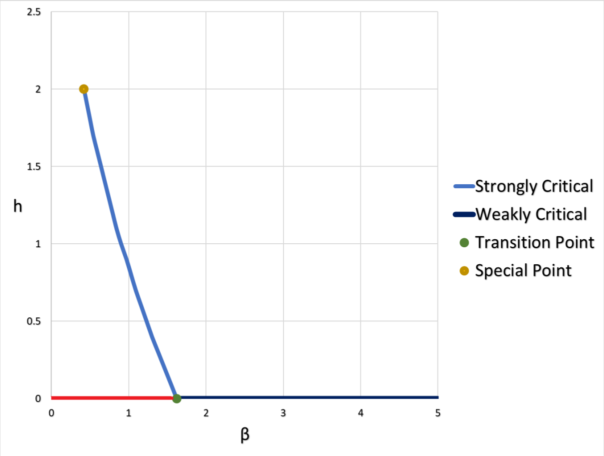

Figure: Special Point: $\hat{\beta}_N$ is $N^{3/4}$ consistent except for $(p, q) = (2, 2), (3, 2)$. $\hat{h}_N$ is $N^{3/4}$ consistent.

Special Case of $p = 4, q = 2$

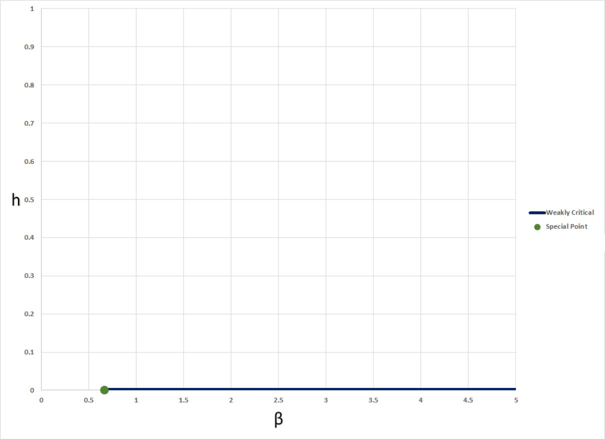

Figure: Special Point: $\hat{\beta}_N$ is $N^{5/6}$ in-consistent. $\hat{h}_N$ is $N^{5/6}$ consistent.

Summary and Discussion

- Phase transitions in the asymptotics of the magnetization vector.

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- For "almost every" points, both $\hat{\beta}_N$ and $\hat{h}_N$ are $\sqrt{N}$-consistent and asymptotically normal.
- At one point, both $\hat{\beta}_N$ and $\hat{h}_N$ can be superefficient, converging at rate $N^{3/4}$ to non-Gaussian distributions.
- There exists a continuous curve in the interior of the parameter plane, on which the MLEs have mixture distributions. The components are normal/half-normal and point masses.
- The curious case of $(p, q) = (4, 2)$, where $\hat{h}_N$ can converge at rate of $N^{5/6}$ to a non-Gaussian distribution.

Thank You!